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Transversal Numbers of Uniform Hypergraphs

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Abstract. The transversal number $\tau(H)$ of a hypergraph H is the minimum cardinality of a set of vertices that intersects all edges of H. For $k \geq 1$ define $c_k = sup\, \tau(H)/(m + n)$, where H ranges over all k-uniform hypergraphs with n vertices and m edges. Applying probabilistic arguments we show that $c_k = (1 + o(1))\frac{\log_e k}{k}$. This settles a problem of Tuza.

1. Introduction

The *transversal number* $\tau(H)$ of a hypergraph H is the minimum cardinality of a set of vertices that intersects all edges of H. Let $H = (V, E)$ be a k-uniform hypergraph with n vertices and m edges. Trivially, there exists a positive constant c_k (which depends only on k) such that $\tau(H) \leq c_k(n + m)$. Tuza [4] proposed the problem of determining or estimating the best possible constants c_k with the above property. Clearly these constants are given by $c_k = sup\, \tau(H)/(m + n)$, where H ranges over all k-uniform hypergraphs with n vertices and m edges. It is easy to check that $c_1 = 1/2$ and $c_2 = 1/3$. Tuza [4] showed that $c_3 = 1/4$ and Feng-Chu Lai and Gerard J. Chang [3] proved that $c_4 = 2/9$, and that $c_k \geq \frac{2}{k + 1 + \lfloor\sqrt{k}\rfloor + \lceil k/\sqrt{k}\rceil}$ for all $k \geq 1$. In [4] the author asks if $c_k = O(1/k)$. In this note we show that this is not the case and prove the following theorem, which determines the asymptotic behaviour of c_k as k grows.

Theorem 1.1. *As k tends to infinity* $c_k = (1 + o(1))\frac{\log k}{k}$.

In the above theorem, and in the rest of the paper, all logarithms are in the natural base e.

There are two parts of the proof of Theorem 1.1. First we show that for any $k > 1$ and for any k uniform hypergraph H with n vertices and m edges, $\tau(H) \leq$

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$\frac{\log k}{k}(n+m)$. Next we establish the existence of a k-uniform hypergraph H with n vertices and m edges satisfying $\tau(H) \geq (1+o(1))\frac{\log k}{k}(n+m)$. Both parts are proved by probabilistic arguments and demonstrate the power of relatively simple probabilistic ideas.

2. An Upper Bound for Transversal Numbers

In this section we prove the following simple proposition, that implies that $c_k \leq \frac{\log k}{k}$.

Proposition 2.1. *Let $H = (V, E)$ be a k-uniform hypergraph with n vertices and m edges, where $k > 1$. Then for any positive real α*

$$\tau(H) \leq n\frac{\alpha \log k}{k} + \frac{m}{k^\alpha}.$$

Proof. Since $\tau(H) \leq n$ we may assume that $\frac{\alpha \log k}{k} \leq 1$, since otherwise there is nothing to prove. Let us pick, randomly and independently, each vertex v of H with probability $p = \frac{\alpha \log k}{k}$. Let $X \subseteq V$ be the (random) set of the vertices picked, and let $F = F_X \subseteq E$ be the set of all edges $e \in E$ that do not intersect X. Clearly, for every fixed $e \in E$, $Prob(e \in F) = (1-p)^k = \left(1 - \frac{\alpha \log k}{k}\right)^k \leq \frac{1}{k^\alpha}$. By the linearity of expectation, we conclude that the expected value of the quantity $|X| + |F|$ is at most $np + \frac{m}{k^\alpha} = n\frac{\alpha \log k}{k} + \frac{m}{k^\alpha}$. Thus, there is at least one choice of a set $X \subseteq V$ so that $|X| + |F_X| \leq n\frac{\alpha \log k}{k} + \frac{m}{k^\alpha}$. By adding to X, arbitrarily, a vertex from each edge in F_X, we obtain a set of at most $|X| + |F_X| \leq n\frac{\alpha \log k}{k} + \frac{m}{k^\alpha}$ vertices of H that intersects all edges. Hence $\tau(H) \leq n\frac{\alpha \log k}{k} + \frac{m}{k^\alpha}$, completing the proof. □

Corollary 2.2. *Suppose $k > 1$ and let H be a k-uniform hypergraph with n vertices and m edges. Then $\tau(H) \leq \frac{\log k}{k}(n+m)$. Therefore $c_k \leq \frac{\log k}{k}$ for all $k > 1$.*

Proof. Since $c_2 = \frac{1}{3} < \frac{\log 2}{2} = 0.3465\ldots$ we may assume that $k \geq 3$. By substituting $\alpha = 1$ in the last Proposition we obtain $\tau(H) \leq n\frac{\log k}{k} + \frac{m}{k} \leq \frac{\log k}{k}(n+m)$, as needed. □

3. Hypergraphs with Relatively Large Transversal Numbers

In this section we assume, whenever it is needed, that k is sufficiently large. Put $n = [k \log k]$ and $m = k$. Let $H = (V, E)$ be a random k-uniform hypergraph on a set V of n vertices, with m (not necessarily distinct) edges, constructed by choosing each of the m edges randomly and independently according to a uniform distribution on the k-subsets of V. We claim that with high probability $\tau(H) > \log^2 k - 10 \log k \cdot \log\log k$. Indeed, let us fix a subset X of cardinality $|X| \le \log^2 k - 10 \log k \log\log k$ of V, and estimate the probability that it is a transversal of H. For each of the m edges e of H, the probability that e does not intersect X satisfies

$$\Pr(e \cap X = \varnothing) = \frac{\binom{n-|X|}{k}}{\binom{n}{k}} \ge \left(\frac{n - |X| - k}{n-k}\right)^k$$

$$\ge \left(\frac{k \log k - \log^2 k + 10 \log k \log\log k - k}{k \log k - k}\right)^k$$

$$= \left(1 - \frac{\log k - 10 \log\log k}{k - k/\log k}\right)^k.$$

To estimate the last quantity observe that for all x, $1 + x \le e^x$ and thus for every $x < 1$, $1 - x \ge e^{-x}(1 - x^2)$. Hence, for all sufficiently large k we have

$$Pr(e \cap X = \varnothing) \ge e^{-((\log k - 10 \log\log k)k)/(k - k/\log k)} \cdot \left[1 - \left(\frac{\log k - 10 \log\log k}{k - k/\log k}\right)^2\right]^k$$

$$= (1 + o(1)) e^{-\log k + 10 \log\log k + 1 + o(1)} = (1 + o(1)) \cdot e \cdot \frac{\log^{10} k}{k} \ge \frac{\log^9 k}{k}.$$

As the $m = k$ edges are chosen independently this implies that the probability that X is a transversal is at most $\left(1 - \frac{\log^9 k}{k}\right)^k \le e^{-\log^9 k}$.

Since there are less than $\sum_{0 \le i \le \log^2 k} \binom{n}{i} < e^{\log^4 k}$ choices for X, this implies that the probability that $\tau(H) \le \log^2 k - 10 \log k \log\log k$ does not exceed $e^{\log^4 k - \log^9 k} = o(1)$. Thus, there is at least one k-uniform hypergraph with $n = [k \log k]$ vertices, and $m = k$ edges satisfying $\tau(H) > \log^2 k - 10 \log k \cdot \log\log k$. We have thus proved;

Proposition 3.1. *For all sufficiently large k*

$$c_k \ge \frac{\log^2 k - 10 \log k \cdot \log\log k}{[k \log k] + k} = \frac{\log k}{k}\left(1 - O\left(\frac{\log\log k}{\log k}\right)\right) = (1 + o(1)) \frac{\log k}{k}.$$

□

Theorem 1.1 is clearly an immediate consequence of Corollary 2.2 and Proposition 3.1.

4. Concluding Remarks and Open Problems

Our construction of k-uniform hypergraphs H with n vertices and m edges for which $\tau(H)/(n+m)$ is $(1+o(1))\frac{\log k}{k}$ is probabilistic. It would be interesting to find an explicit construction of such hypergraphs. At the moment we are unable to give such a construction, but we can construct explicitly for infinitely many values of k, k-uniform hypergraphs H with $2k+1$ vertices, $2k+1$ edges and with $\tau(H) \geq (\frac{1}{2}+o(1)) \log k$. Indeed, let $q = 2k+1$ be an odd prime power, and let H be the hypergraph whose vertices are all the elements of the finite field $GF(q)$, and whose edges are the following q edges; for each $y \in GF(q)$, e_y is the edge $e_y = \{x \in GF(q): x - y \text{ is not a square in } GF(q)\}$. Clearly H has $n = 2k+1$ vertices and $m = 2k+1$ edges, and it is k uniform. The fact that $\tau(H) \geq (\frac{1}{2} - o(1)) \log k$ follows easily from the well known method of applying known results about character sums to derive the pseudo-random properties of quadratic tournaments and Paley Graphs, (as in, e.g. [2], [1].) We omit the details.

It would be extremely interesting to determine precisely the value of c_k for every k. The considerable effort made in [3] to show that $c_4 = 2/9$ suggests that this may be difficult.

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